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Some automorphisms of Generalized Kac-Moody algebras

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1. Introduction

In this paper we consider some algebraic structures associated to a class of outer automorphisms of generalized Kac-Moody (GKM) algebras. These structures have recently been introduced in [2] for a smaller class of outer automorphisms in the case of ordinary Kac-Moody algebras with symmetrizable Cartan matrices.

A GKM algebra $G = G(A)$ is essentially described by its Cartan matrix, $A = (a_{ij})_{i,j\in I}$; the index set I can be either a finite or a countably infinite set. For any permutation $\dot\omega$ of the set I which has finite order and leaves the Cartan matrix invariant, we find a family of outer automorphisms ω of the GKM algebra $G(A)$ which preserve the Cartan decomposition.

Such an outer automorphism gives rise to a linear bijection τ_ω of G-modules, obeying the ω-twining property, i.e. if V is a G-module, then

$$\tau_\omega(xv) = (\omega^{-1}x)\tau_\omega(v)$$

for all $x\in G$ and all $v\in V$. Thus in general τ_ω is not a homomorphism of G-modules, but some sort of "twisted homomorphism". Furthermore τ_ω maps highest weight G-modules to highest weight G-modules, though the image and pre-image are not always isomorphic.

In applications in conformal field theory, one is particularly interested in those highest weight modules which are mapped to themselves. The dual map ω^* of the restriction of ω to a Cartan subalgebra H of G is a bijection of H^*, the dual of H. For a highest weight G-module $V(\Lambda)$ of highest weight Λ we have $\tau_\omega(V(\Lambda)) = V(\Lambda)$ if and only if $\omega^*(\Lambda) = \Lambda$. A convenient tool to keep track of some properties of such a linear map is the twining character of $V(\Lambda)$, defined as in [2] as the formal sum

$$(ch\,V)^\omega = \sum_{\lambda\le\Lambda} m_\lambda^\omega\, e(\lambda)\,,$$

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where

$$m_\lambda^\omega = \begin{cases} 0\,, & \text{if } \omega^*(\lambda) \neq \lambda\,; \\ \mathrm{tr}(\tau_\omega|_{V_\lambda})\,, & \text{if } \omega^*(\lambda) = \lambda\,. \end{cases}$$

The main result of this paper is an explicit formula for the twining character of Verma and irreducible highest weight G-modules. This formula shows that the twining characters can be described in terms of the characters of highest weight modules of some other GKM algebra which depends on $G = G(A)$ and $\dot\omega$, the so-called orbit Lie algebra. In this paper we show that the 'linking condition' that had to be imposed in [2] is not needed, and that in particular this result applies to all Kac-Moody algebras with symmetrizable Cartan matrices. In the case of affine Lie algebras, this result has allowed for the solution of two long-standing problems in conformal field theory: the resolution of field identification fixed points in diagonal coset conformal field theories (see [3]) and the resolution of fixed points in integer spin simple current modular invariants (see [4]).

In Section 2, after recalling the definition of a GKM algebra, we introduce the notion of an orbit Lie algebra and a twining character. In Section 3 we state and prove our main theorem, Theorem 3.1, which asserts that twining characters are described by ordinary characters of the orbit Lie algebra, for a particular type of automorphisms of G which just permute the generators associated to simple roots. As a by-product, we associate in Proposition 3.3 to the permutation $\dot\omega$ an interesting subgroup $\hat W$ of the Weyl group W of a GKM algebra, which is again a Coxeter group. In Section 4 we extend the Theorem to the whole class of outer automorphisms associated to a given finite order permutation $\dot\omega$ of the index set I, leaving the Cartan matrix invariant.

Apart from the extension to arbitrary GKM algebras and to a larger class of automorphisms, the present paper improves the treatment in [2] inasfar as the description of $\hat W$ and the analysis of the cases with $\sum_{l=0}^{N_i-1} a_{i,\dot\omega^l i} \le 0$ are concerned. The corresponding statements, which previously had to be verified by detailed explicit calculations (see e.g. the appendix of [2]), are now immediate consequences of our general results.

2. Definitions and elementary properties

We first remind the reader of the definition of a Generalized Kac-Moody (GKM) algebra and of some of its properties (see [1] or [5] for details). All vector spaces considered are complex. Let I be either a finite or a countably infinite set. For simplicity of notation, we identify I with $\{1, 2, \ldots, n\}$ or $\mathbb{Z}_+$.

Let $A = (a_{ij})_{i,j \in I}$ be a matrix with real entries defined as follows:

(i) $a_{ij} \le 0$ if $i \neq j$;

(ii) $\frac{2a_{ij}}{a_{ii}} \in \mathbb{Z}$ if $a_{ii} > 0$;

(iii) if $a_{ij} = 0$, then $a_{ji} = 0$;

(iv) there exists a diagonal matrix $D = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_n)$, with $\epsilon_i \in \mathbb{R}$ and $\epsilon_i > 0$ for all i, such that DA is symmetric.

A matrix satisfying condition (iv) is said to be symmetrizable.

Let H be an abelian Lie algebra of dimension greater or equal to n. Let $h_1, \dots, h_n$ be linearly independent elements of H. Define α_j in H^*, the dual of H, to be such that $\alpha_j(h_i) = a_{ij}$.

The GKM algebra $G = G(A)$ with Cartan matrix A and Cartan subalgebra H is a Lie algebra generated by e_i, f_i, $i \in I$, and H, with the following defining relations:

$$
\begin{aligned}
&[e_i, f_j] = \delta_{ij} h_i \,, \\
&[h, e_i] = \alpha_i(h) e_i \,, \\
&[h, f_i] = -\alpha_i(h) f_i \,, \\
&(\mathrm{ad}\, e_i)^{1-2a_{ij}/a_{ii}} e_j = 0 = (\mathrm{ad}\, f_i)^{1-2a_{ij}/a_{ii}} f_j \quad \text{if } a_{ii} > 0 \,, \\
&[e_i, e_j] = 0 = [f_i, f_j] \quad \text{if } a_{ij} = 0 \,.
\end{aligned}
$$

Remarks. 1. For any $n \times n$ diagonal matrix D' with real positive entries, $G(D'A)$ is a GKM algebra isomorphic to $G(A)$. The resulting generators are scalar multiples of the above ones, and so the simple roots α_j remain unchanged. If we take D' to be $\mathrm{diag}(\epsilon'_1, \epsilon'_2, \dots, \epsilon'_n)$, where

$$
\epsilon'_i = \begin{cases} 2(a_{ii})^{-1}, & \text{if } a_{ii} > 0 \,, \\ 1 \,, & \text{otherwise} \,, \end{cases}
$$

then all the diagonal positive entries of the matrix $D'A$ are equal to 2, and we get a Cartan matrix as defined in [5].
2. In [2], h_i, and α_j are defined to be so that $\alpha_j(h_i) = a_{ji}$. In this paper we have taken the transpose in order to follow the convention in [5].

The elements $h_1, h_2, \dots, h_n$ form a basis for $H \cap [G, G]$, and there is a subalgebra C consisting of commuting derivations of G such that $H = H \cap [G, G] \oplus C$.
There exists a bilinear form $(.\,,.)$ on H defined by:

$$
(h_i, h) = \epsilon_i^{-1} \alpha_i(h) \,, \quad \text{and} \quad (h, h') = 0 \,, \quad h, h' \in C \,,
$$

where as above the matrix $\mathrm{diag}(\epsilon_1, \dots, \epsilon_n)A$ is symmetric.
Therefore $(h_i, h_j) = \epsilon_j^{-1} a_{ij}$. This form extends uniquely to a bilinear, symmetric, invariant form on G, whose kernel is contained in H. When the form is non-degenerate, it induces a bilinear form on H^*, which we also denote by $(.\,,.)$. In particular,

$$
(\alpha_i, \alpha_j) = \epsilon_i a_{ij} \,.
$$

Note that H (and hence G) can always be extended by adding outer derivations of G having the e_i, f_i as eigenvectors so as to make the form non-degenerate. When $(.\,,.)$ is non-degenerate on H, the Cartan decomposition holds for the GKM algebra G:

$$
G = \Big(\bigoplus_{\alpha \in \Delta^+} G_{-\alpha} \Big) \oplus H \oplus \Big(\bigoplus_{\alpha \in \Delta^+} G_{\alpha} \Big) \,,
$$

where $G_\alpha = \{x \in G \mid [h, x] = \alpha(h)x,\ h \in H\}$, and Δ^+ is the set of positive roots of G (i.e. $\alpha \in H^*$ is a positive root if $G_\alpha \neq 0$, and α is a sum of simple roots).

If $a_{ii} > 0$, the simple root α_i is called real. Set

$$I_{\mathrm{r}} := \{i \in I \mid a_{ii} > 0\}\,.$$

The Weyl group W is generated by the reflections r_i, $i \in I_{\mathrm{r}}$, acting on H^*. A root is said to be real if it is conjugate to a real simple root under the action of W, and imaginary otherwise. The group W is a Coxeter group (see Proposition 3.13 in [5]). Recall that a Coxeter group is a group of the following type:

$$< x_1, x_2, \dots, x_n \mid x_i^2 = 1;\ (x_i x_j)^{m_{ij}} = 1\ (i, j = 1, 2, \dots, n,\ i \neq j) >,$$

where the m_{ij} are positive integers or ∞. Set m_{ij} to be the order of $(r_i r_j)$ for $i, j \in I_{\mathrm{r}}$ $(i \neq j)$. These orders are given by the following table, which we call T (see [5]):

$\frac{2a_{ij}}{a_{ii}}\frac{2a_{ji}}{a_{jj}}$	0	1	2	3	≥ 4
m_{ij}	2	3	4	6	∞

In the rest of this paper $G = G(A)$ will denote a GKM algebra, with non-degenerate bilinear form $(.\,,.)$. Without loss of generality, we assume that the Cartan matrix A is symmetric.

Choose a bijection $\dot\omega\colon I \mapsto I$ of finite order which keeps the Cartan matrix fixed, i.e. $a_{\dot\omega i, \dot\omega j} = a_{i,j}$ for all $i, j \in I$.

If the Dynkin diagram of G is defined to be the Dynkin diagram of the GKM subalgebra of G generated by e_i, f_i for all $i \in I_{\mathrm{r}}$, then $\dot\omega$ restricts to a bijection of the Dynkin diagram. (Note that the number of bonds linking node i and j is $\max\{\frac{2|a_{ij}|}{a_{ii}}, \frac{2|a_{ji}|}{a_{jj}}\}$).

Let N be the order of $\dot\omega$ and N_i the length of the $\dot\omega$-orbit of i in I.

By the same arguments that were given in §3.2 of [2] for Kac-Moody algebras, $\dot\omega$ induces an outer automorphism ω of G (the details of its action on the outer derivations in H are given in [2]). In particular,

$$\omega e_i = e_{\dot\omega i}\,, \quad \omega f_i = f_{\dot\omega i}\,, \quad \omega h_i = h_{\dot\omega i}\,, \quad \text{for all}\ \ i \in I\,.$$

The automorphism ω preserves the Cartan decomposition. Let $\zeta \in \mathbb{C}$ be a primitive Nth root of unity. Then the eigenvalues of the restriction $\omega|_H$ of ω to H are contained in $\{\zeta^l \mid l = 0, 1, \dots, N-1\}$. Since $\omega|_H$ has finite order, H is the direct sum of its eigenspaces. Let H^l denote the eigenspace corresponding to eigenvalue ζ^l.

We choose a set of representatives from each $\dot\omega$-orbit:

$$\hat I := \{i \in I \mid i \leq \dot\omega^l i,\ \forall l \in \mathbb{Z}\}\,.$$

Some of these orbits play a major role, so we also introduce the following subset of $\hat I$:

$$\breve I := \{i \in \hat I \mid \sum_{l=0}^{N_i - 1} a_{i,\dot\omega^l i} \leq 0 \Rightarrow \sum_{l=0}^{N_i - 1} a_{i,\dot\omega^l i} = a_{ii}\}\,.$$

For $i\in\hat{I}$ define

$$s_i := \begin{cases} a_{ii}/\sum_{l=0}^{N_i-1} a_{i,\dot{\omega}^l i}\,, & \text{if } i\in\breve{I} \text{ and } a_{ii}\neq 0\,, \\ 1\,, & \text{otherwise}\,. \end{cases}$$

Remarks. 1. Suppose that $i\in\breve{I}$. The definition of $\breve{I}$ implies that if α_i is imaginary, then $s_i=1$, and if α_i is real, then we only have two possibilities: either $s_i=1$ and for all integers $1\le l\le N_i-1$, $a_{i,\dot{\omega}^l i}=0$, or else $s_i=2$ and there is a unique integer $1\le l\le N_i-1$ such that $a_{i,\dot{\omega}^l i}\neq 0$. In the latter case, $\frac{2a_{i,\dot{\omega}^l i}}{a_{ii}}=-1$. So we can deduce that N_i is even, and $l=\frac{N_i}{2}$. Hence when $s_i=1$, the orbit of i in the Dynkin diagram of G is totally disconnected, i.e. of type $A_1\times\cdots\times A_1$ (where A_1 appears N_i times); and when $s_i=2$, the orbit of i is of type $A_2\times\cdots\times A_2$ (where A_2 appears $\frac{N_i}{2}$ times).
2. If G is a Kac-Moody algebra and $\dot{\omega}$ fulfills the linking condition of [2], then $\breve{I}=\hat{I}$. We will not need to impose this condition.

Define the matrix $\hat{A}=(\hat{a}_{ij})_{i,j\in\hat{I}}$ to be as follows:

$$\hat{a}_{ij} := s_j \sum_{l=0}^{N_j-1} a_{i,\dot{\omega}^l j}.$$

Lemma 2.1. *The matrix $\hat{A}$ satisfies conditions* (i), (ii), (iii), *and* (iv) *of the Cartan matrix of a GKM algebra.*

Proof. Suppose $i\neq j\in\hat{I}$. Then $\hat{a}_{ij}\le 0$ since for all integers l, $\dot{\omega}^l j\neq i$ as i and j are not in the same $\dot{\omega}$-orbit. Suppose further that $\hat{a}_{ii}>0$. Then $\hat{a}_{ii}=a_{ii}$, so that $\frac{2\hat{a}_{ij}}{\hat{a}_{ii}}$ is an integer since s_j is an integer for all $j\in\hat{I}$.
If $\hat{a}_{ij}=0$, then $0=\sum_{l=0}^{N_j-1} a_{i,\dot{\omega}^l j}=\frac{N_j}{N_i}\sum_{l=0}^{N_i-1} a_{j,\dot{\omega}^l i}$, so that $\hat{a}_{ji}=0$.
Let $\hat{D}=\operatorname{diag}(N_i s_i)_{i\in\hat{I}}$. Then straightforward calculations show that $\hat{D}\hat{A}$ is symmetric. This completes the proof. $\square$

Therefore there is a GKM algebra, which we call $\hat{G}$, with Cartan matrix $\hat{A}$, and such that the bilinear form induced by $\hat{A}$ is non-degenerate on its Cartan subalgebra $\hat{H}$. We let $\hat{e}_i$, and $\hat{f}_i$ denote its other generators. Set $\hat{h}_i=[\hat{e}_i,\hat{f}_i]\,,\ i\in\hat{I}$.

Remarks. 1. If $i\in\hat{I}-\breve{I}$, then $a_{ii}/\sum_{l=0}^{N_i-1} a_{i,\dot{\omega}^l i}$ is not an integer when $a_{ii}\le 0$, and it is non-positive when $a_{ii}>0$. Therefore if $\hat{I}\neq\breve{I}$, then the matrix with entries $\left(\frac{a_{jj}}{\sum_{l=0}^{N_j-1} a_{j,\dot{\omega}^l j}}\right)\sum_{l=0}^{N_j-1} a_{i\dot{\omega}^l j}$ is not the Cartan matrix of a GKM algebra.
2. The elements of G fixed by ω form a GKM subalgebra of G (see [1] for the proof). This fixed point subalgebra has a GKM subalgebra whose Cartan matrix has (i,j)-th entry equal to $\sum_{l=0}^{N_i-1} a_{\dot{\omega}^l i,j}$. However, $\hat{G}$ is not in general isomorphic to a subalgebra of G.

We are now ready to define the *orbit Lie algebra* associated to the automorphism ω. To do so, we have to use the subset $\breve{I}$ rather than $\hat{I}$.

Definition 2.1. *The orbit Lie algebra associated to the bijection $\dot\omega$ of the Cartan matrix A, or equivalently to the automorphism ω of G, is defined to be the Lie subalgebra $\breve G$ of $\hat G$ generated by $\hat e_i$, $\hat f_i$ for $i \in \breve I$, and $\hat H$.*

Lemma 2.1 implies that $\breve G$ is a GKM algebra with Cartan matrix

$$\breve A = (\hat a_{ij})_{i,j\in \breve I}\,.$$

Remarks. 1. The GKM subalgebra $\breve G$ of $\hat G$ is also in general not isomorphic to a subalgebra of G.
2. The set $\breve I$ may be empty, in which case $\breve G = 0$.
3. It can be shown that if G is of finite type, then so is the orbit Lie algebra $\breve G$; and if G is of affine (resp. indefinite) type, then $\breve G$ is either trivial, or also of affine (resp. indefinite) type (see [2]).

We next define a linear map $P_\omega\colon\ H^0 \cap [G,G] \to \hat H$ as follows:

$$P_\omega\Big(\sum_{l=0}^{N_i-1} h_{\dot\omega^l i}\Big) = N_i \hat h_i\,.$$

Lemma 2.2. *For all $h, h' \in [G,G]\cap H^0$, $(h,h') = (P_\omega(h), P_\omega(h'))$.*

Proof. Let i be in $\hat I$. Since $N_i = N_{\dot\omega^l i}$ for all integers l,

$$\begin{aligned}(\sum_{l=0}^{N_i-1} h_{\dot\omega^l i}, \sum_{l=0}^{N_j-1} h_{\dot\omega^l j}) &= N_i \sum_{l=0}^{N_j-1} a_{i,\dot\omega^l j} = N_i s_j^{-1}\hat a_{ij} \\ &= (N_i\hat h_i, N_j \hat h_j) = (P_\omega \sum_{l=0}^{N_i-1} h_{\dot\omega^l i}, P_\omega \sum_{l=0}^{N_j-1} h_{\dot\omega^l j})\,.\end{aligned}$$

The result follows by linearity. $\square$

Provided we choose $\hat H$ to have the right number of outer derivations, this map can be extended to the outer derivations contained in H^0 so as to give an isomorphism $H^0 \to \hat H$, in such a way that Lemma 2.2 holds for all h, h' in H^0 (the proof is the same as in §3.3 of [2], where this is shown for Kac-Moody algebras). For simplicity of notation, we also call this isomorphism P_ω.

The automorphism ω induces a dual map ω^* on H^*, namely:

$$(\omega^*\beta)(h) = \beta(\omega h)\,, \quad \text{for} \quad \beta\in H^*,\ h\in H.$$

In particular,

$$\omega^*(\alpha_i) = \alpha_{\dot\omega^{-1} i}\,,$$

since $(\omega^*(\alpha_i))(h) = \alpha_i(\omega(h)) = (h_i, \omega(h)) = (h_{\dot\omega^{-1}i}, h) = \alpha_{\dot\omega^{-1}i}(h)$ for all $h\in H$.

This bijection has the same eigenvalues as $\omega|_H$, and so $(H^*)^l$ will denote the eigenspace corresponding to eigenvalue ζ^l.

Since its restriction to H^0 is non-degenerate, the bilinear form gives rise to a bijection between $(H^0)^*$ and $(H^*)^0$. Hence P_ω induces a dual map $P_\omega^*\colon \hat{H}^* \to (H^*)^0$. By definition $\lambda \in (H^*)^0$ if and only if $\omega^*(\lambda) = \lambda$. Such weights will be called *symmetric weights.* Set

$$\beta_i := \sum_{l=0}^{N_i-1} \alpha_{\dot{\omega}^l i}\,, \quad \text{for each} \quad i \in I.$$

In particular the following holds:

Lemma 2.3.
(i) *For all* $i \in \hat{I}$, $P_\omega^*(\hat{\alpha}_i) = s_i\beta_i$ *; and*
(ii) *for all* $\lambda, \mu \in (H^*)^0$, $(\lambda, \mu) = (P_\omega^{*-1}(\lambda), P_\omega^{*-1}(\mu))$.

Proof. For all $h \in H^0$, Lemma 2.2 implies that

$$\hat{\alpha}_i(P_\omega(h)) = N_i s_i\, (\hat{h}_i, P_\omega(h)) = s_i\, (\sum_{l=0}^{N_i-1} h_{\dot{\omega}^l i}, h)\,,$$

so that (i) holds.
(ii) follows directly from Lemma 2.2 and the definition of P_ω^*. □

We next define the *twining character* of a highest weight G-module. We first need to associate a representation R^ω to a given representation R of the GKM algebra G.

Definition 2.2. *Let* V *be a* G*-module and* $R\colon G \to \mathrm{gl}(V)$ *the corresponding representation. Define* R^ω *to be the representation* $G \to \mathrm{gl}(V)$ *such that* $R^\omega(x) = R(\omega(x))$ *for all* $x \in G$.

Let $R_\Lambda\colon\, G \to \mathrm{gl}(V(\Lambda))$ be a highest weight G-representation of highest weight Λ in H^*. Then $(R_\Lambda)^\omega$ is a highest weight representation of highest weight $\omega^*(\Lambda)$, $R_{\omega^*(\Lambda)}:\, G \to \mathrm{gl}(V(\omega^*(\Lambda)))$, since ω preserves the Cartan decomposition. Thus the automorphism ω induces a bijection of G-modules $\tau_\omega:\; V(\Lambda) \to V(\omega^*(\Lambda))$ which satisfies the ω*-twining property*, i.e.

$$\tau_\omega(R_\Lambda(x)v) = R_{\omega^*(\Lambda)}(\omega^{-1}x)\tau_\omega(v) \quad \text{for all } v \in V(\Lambda),\; x \in G\,.$$

Remarks. When $\omega^*(\Lambda) \neq \Lambda$, the representations R_Λ and $(R_\Lambda)^\omega$ are not isomorphic. The bijection τ_ω is a linear map, but not in general an isomorphism of G-modules, even if $\omega^*(\Lambda) = \Lambda$.

Denote by $V_\lambda := \{v \in V \mid hv = \lambda(h)v,\, h \in H\}$ the weight space of V of weight λ. The bijection τ_ω maps V_λ onto $V_{\omega^*(\lambda)}$ In particular if R corresponds to the Verma (resp.

irreducible highest weight) module $M(\Lambda)$ (resp. $L(\Lambda)$), then R^ω corresponds to the Verma (resp. irreducible highest weight) module $M(\omega^*(\Lambda))$ (resp. $L(\omega^*(\Lambda))$).

In this paper, we study the case when Λ is a symmetric weight (i.e. $\omega^*(\Lambda) = \Lambda$), so that τ_ω maps $M(\Lambda)$ (respectively $L(\Lambda)$) to itself. In the physics literature, such weights are called *fixed points* (see [6]).

The ordinary character $\mathrm{ch}\, V$ of a highest weight G-module V is the formal sum

$$\mathrm{ch}\, V := \sum_\lambda (\dim V_\lambda)\, e(\lambda)\,.$$

Replacing the formal exponential by the exponential function, this gives a complex valued function

$$\mathrm{ch}\, V(h) = \sum_\lambda (\dim V_\lambda)\, \mathrm{e}^{\lambda(h)} = \mathrm{tr}_V \mathrm{e}^h\,,$$

defined on the set $Y(V)$ of elements $h \in H$ such that the series converges absolutely.

Let $\rho \in H^*$ be a Weyl vector for G, i.e. $(\rho, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i)$ for all $i \in I$. Such a vector exists since by assumption $(.,.)$ is non-degenerate on H. For $w \in W$, let $\epsilon(w) = (-1)^l$, where l is the minimal number of simple reflections r_i needed to write w. Let $S_\Lambda = e(\Lambda + \rho) \sum_\beta \epsilon(\beta) e(-\beta)$, where $\epsilon(\beta) = (-1)^m$ if β is a sum of m distinct pairwise orthogonal imaginary simple roots, orthogonal to Λ, and $\epsilon(\beta) = 0$ otherwise. Borcherds showed that if $\Lambda \in H^*$, $(\Lambda, \alpha_i) \geq 0$ for all $i \in I$, and $2(\Lambda, \alpha_i)/(\alpha_i, \alpha_i) \in \mathbb{Z}$ for all real simple roots α_i of G, then the irreducible module $L(\Lambda)$ of highest weight Λ has character

$$\mathrm{ch}\, L(\Lambda) = \sum_{w \in W} \epsilon(w) w(S_\Lambda)/e(\rho) \prod_{\alpha \in \Delta^+} (1 - e(-\alpha))^{\mathrm{mult}\alpha}\,.$$

(For details of the proof of the character formula, see [1] or [5].)

Definition 2.3. *Let $\Lambda \in H^*$ be a symmetric weight. We define the twining character for the highest weight representation R_Λ on $V(\Lambda)$ to be the following complex valued function defined on $Y(V)$: $(\mathrm{ch}\, V(\Lambda))^\omega(h) = \mathrm{tr}_V \tau_\omega \mathrm{e}^{R_\Lambda(h)}$.*

Note that since the twining character is bounded by the ordinary character, it is absolutely convergent on $Y(V)$. Equivalently the twining character is the formal sum

$$(\mathrm{ch}\, V(\Lambda))^\omega = \sum_{\lambda \leq \Lambda} m_\lambda^\omega\, e(\lambda)\,,$$

where

$$m_\lambda^\omega = \begin{cases} 0\,, & \text{if } \omega^*(\lambda) \neq \lambda\,; \\ \mathrm{tr}(\tau_\omega|_{V_\lambda})\,, & \text{if } \omega^*(\lambda) = \lambda\,. \end{cases}$$

Let $\mathcal{V}_\Lambda$ (resp. Ψ_Λ) denote the ordinary character of the Verma (resp. irreducible) G-module of highest weight Λ, and $\breve{\mathcal{V}}_{\breve{\Lambda}}$ (resp. $\breve{\Psi}_{\breve{\Lambda}}$) denote the ordinary character of the Verma (resp. irreducible) $\breve{G}$-module of highest weight $\breve{\Lambda}$.

Remarks. A weight Λ in H^* is said to be integrable if $(\Lambda, \alpha_i) \geq 0$ for all $i \in I$ and $\frac{2(\Lambda,\alpha_i)}{(\alpha_i,\alpha_i)}$ is an integer for all real simple roots α_i. A G-module V is called integrable if f_i and e_i act locally nilpotently for all $i \in I$ such that $a_{ii} > 0$. An irreducible G-module of highest weight Λ is unitarizable if and only if Λ is integrable. If all the simple roots of G are real, i.e. G is a Kac-Moody algebra, then an irreducible G-module of highest weight Λ is integrable if and only if it is unitarizable (see §3, 10, and 11 in [5] for details).

3. Twining characters and orbit Lie algebras

We now state the main result of this paper.

Theorem 3.1. *Let $\Lambda \in H^*$ be a symmetric weight, i.e. $\omega^*(\Lambda) = \Lambda$. The twining character of the Verma G-module of highest weight Λ coincides with the ordinary character of the Verma $\breve{G}$-module of highest weight $P_\omega^{*-1}(\Lambda)$:*

$$P_\omega^{*-1}(\mathcal{V}_\Lambda)^\omega = \breve{\mathcal{V}}_{P_\omega^{*-1}(\Lambda)}.$$

If, moreover, Λ is integrable, then the twining character of the irreducible G-module of highest weight Λ coincides with the ordinary character of the irreducible $\breve{G}$-module of integrable highest weight $P_\omega^{*-1}(\Lambda)$:

$$P_\omega^{*-1}(\Psi_\Lambda)^\omega = \breve{\Psi}_{P_\omega^{*-1}(\Lambda)}.$$

In order to prove this Theorem, we first need a few more results. Any Weyl vector ρ of G satisfies $(\omega^*(\rho), \alpha_i) = (\rho, \alpha_i)$ for all $i \in I$, and hence we can choose ρ to be a symmetric weight.

Lemma 3.2. *The weight $\breve{\rho} = P^{*-1}(\rho)$ is a Weyl vector in $\hat{H}^*$ for $\breve{G}$.*

Proof. Lemma 2.3 implies that if $i \in \breve{I}$ and $a_{ii} \neq 0$, then

$$(\breve{\rho}, \hat{\alpha}_i) = s_i(\rho, \beta_i) = \frac{1}{2} s_i N_i a_{ii} = \frac{1}{2} s_i^2 N_i \sum_{l=0}^{N_i-1} a_{i,\dot{\omega}^l i} = \frac{1}{2} (\hat{\alpha}_i, \hat{\alpha}_i) \, .$$

The penultimate equality follows from the definition of s_i.
If $i \in \breve{I}$ and $a_{ii} = 0$, then $a_{i,\dot{\omega}^l i} = 0$ for all integers l, so that $(\breve{\rho}, \hat{\alpha}_i) = 0 = \frac{1}{2}(\hat{\alpha}_i, \hat{\alpha}_i)$. $\square$

Remarks. 1. The proof of the previous Lemma shows that in order for $\breve{\rho}$ to be a Weyl vector for $\breve{G}$, we need to scale the numbers $\sum_{l=0}^{N_j-1} a_{i,\dot{\omega}^l j}$ by s_j to define the Cartan matrix $\breve{A}$.
2. For $i \in \hat{I} - \breve{I}$, one has $(\breve{\rho}, \hat{\alpha}_i) \neq \frac{1}{2}(\hat{\alpha}_i, \hat{\alpha}_i)$, so that when $\hat{I} \neq \breve{I}$, $\breve{\rho}$ is *not* a Weyl vector for the bigger GKM algebra $\hat{G}$. It is not possible to scale the Cartan matrix $\hat{A}$ in such a way that on the one hand, the resultant matrix remains the Cartan matrix of a GKM algebra, and on the other hand, the resultant vector $\breve{\rho}$ is a Weyl vector for the corresponding GKM algebra.

Let W be the Weyl group of G, and $\breve{W}$ the Weyl group of $\breve{G}$. Note that $\hat{a}_{ii} > 0$ for $i \in \hat{I}$ implies that $i \in \breve{I}$. Since $\hat{G}$ and $\breve{G}$ have the same Cartan subalgebra, $\breve{W}$ is therefore also the Weyl group of $\hat{G}$. Let

$$\breve{I}_{\rm r} := \{i \in \hat{I} \mid \hat{a}_{ii} > 0\}\,,$$

and for $i \in \breve{I}_{\rm r}$ let $\hat{r}_i$ denote the reflections corresponding to the simple real roots of $\hat{G}$ (or equivalently $\breve{G}$). Define

$$\hat{W} := \{w \in W \mid w\omega^* = \omega^* w\}$$

to be the set of all elements in the Weyl group W of G commuting with the bijection ω^* of H^*. This is a subgroup of W.

If $s_i = 2$, then the orbit of i in the Dynkin diagram of G is the product of $\frac{N_i}{2}$ copies of the Dynkin diagram of A_2. Then $\{\dot{\omega}^l i, \dot{\omega}^{l+\frac{N_i}{2}} i\}$ are the connected components of the orbit of i. For each $i \in \breve{I}_{\rm r}$ (i.e. $\hat{a}_{ii} > 0$), define

$$w_i := \begin{cases} r_i r_{\dot{\omega} i} \cdots r_{\dot{\omega}^{N_i-1} i}\,, & \text{if } s_i = 1\,; \\ \prod_{l=1}^{N_i/2} r_{\dot{\omega}^l i} r_{\dot{\omega}^{l+N_i/2} i} r_{\dot{\omega}^l i}\,, & \text{if } s_i = 2\,. \end{cases}$$

As in §5.1 of [2], it can be shown that the elements w_i are in $\hat{W}$, and that for symmetric weights $\lambda \in H^*$,

$$w_i(\lambda) = \lambda - \frac{2 s_i\,(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \sum_{l=0}^{N_i-1} \alpha_{\dot{\omega}^l i}\,. \tag{1}$$

In fact the elements w_i generate the group $\hat{W}$:

Proposition 3.3. $\hat{W} = < w_i \mid i \in \breve{I}_{\rm r} >$.

Proof. Set $\tilde{W} := < w_i \mid i \in \breve{I}_{\rm r} >$. Let $\Lambda \in H^*$ be an integrable symmetric weight such that $(\Lambda, \alpha_i) > 0$ for all $i \in I$. Such weights exist since $(\,.\,,.\,)$ is non-degenerate on H^*. Let $\lambda \le \Lambda$ be a symmetric weight.

We claim that if $i \in I_{\rm r}$ and $\beta_i = \sum_{l=0}^{N_i-1} \alpha_{\dot{\omega}^l i}$ satisfies $(\beta_i, \beta_i) \le 0$, then $(\lambda, \alpha_i) \ge 0$.
Since both Λ and λ are symmetric, $\Lambda - \lambda = \sum_{i \in \hat{I}} k_i \beta_i$, where $k_i \ge 0$ for each $i \in \hat{I}$. Since $(\beta_i, \beta_i) = N_i(\beta_i, \alpha_i)$ and $(\beta_j, \alpha_i) \le 0$ for all $j \ne i$, our claim follows.

Since Λ is integrable, $w(\lambda) \le \Lambda$ for all w in the Weyl group W (see §3 in [5]). Let $w \in \tilde{W}$ be such that the height of $\Lambda - w(\lambda)$ is minimal, i.e. $\mathrm{ht}(\Lambda - w(\lambda)) \le \mathrm{ht}(\Lambda - w'(\lambda))$ for all $w' \in \tilde{W}$.

We claim that $w(\lambda)$ is in the positive Weyl chamber, i.e. for all $i \in I$ such that $a_{ii} > 0$, $(w(\lambda), \alpha_i) \ge 0$.
Assume this is false. Since w commutes with ω, $w(\lambda)$ is symmetric. Hence the above argument implies that for all $j \in \hat{I}_{\rm r} - \breve{I}_{\rm r}$, $(w(\lambda), \alpha_j) \ge 0$. Thus there is some $i \in \breve{I}_{\rm r}$ such that $(w(\lambda), \alpha_i) < 0$. From (1) we get $w_i w(\lambda) = w(\lambda) - s_i \frac{2(w(\lambda), \alpha_i)}{(\alpha_i, \alpha_i)} \beta_i$, so that $\mathrm{ht}(\Lambda - w_i w(\lambda)) < \mathrm{ht}(\Lambda - w(\lambda))$, contradicting the definition of w.

Let w' be in $\hat{W}$. Then $w'(\Lambda)$ is a symmetric weight. So from the above, there is some $\tilde{w}$ in $\tilde{W}$ such that $\tilde{w} w'(\Lambda)$ is in the positive Weyl chamber. Since the W-orbit of Λ intersects the

positive Weyl chamber at a unique point, we can deduce that $\tilde{w}w'(\Lambda) = \Lambda$. Furthermore by definition $(\Lambda, \alpha_i) \neq 0$ for all $i \in I$. Hence $\tilde{w}w' = 1$ (see Proposition 3.12 in [5]), so that $w' \in \tilde{W}$, and hence $\tilde{W} = \hat{W}$. $\square$

The next result shows that $\hat{W}$ is a Coxeter group.

Corollary 3.4. *The subgroup $\hat{W}$ of W is isomorphic to the Weyl group $\breve{W}$ of $\breve{G}$.*

Proof. We first show that for all $i, j \in \breve{I}_{\mathrm{r}}$, $(w_i w_j)^{\breve{m}_{ij}} = 1$, where the exponents $\breve{m}_{ij}$ are given by table T (changing a_{ij}, a_{ji}, a_{ii}, and a_{jj} to $\hat{a}_{ij}$, $\hat{a}_{ji}$, $\hat{a}_{ii}$, and $\hat{a}_{jj}$ respectively, in the table). From Lemma 2.3 and $\hat{D} = \mathrm{diag}(N_i s_i)$, for all $i \in \breve{I}_{\mathrm{r}}$ and all $\hat{\lambda} \in (\hat{H})^*$ we have

$$\frac{(\hat{\lambda}, \hat{\alpha}_i)}{(\hat{\alpha}_i, \hat{\alpha}_i)} = \frac{(P^*_\omega(\hat{\lambda}), \beta_i)}{N_i a_{ii}} = \frac{N_i\,(P^*_\omega(\hat{\lambda}), \alpha_i)}{N_i a_{ii}}$$

since $P^*_\omega(\hat{\lambda})$ is symmetric and $P^*_\omega(\hat{\alpha}_i) = s_i \beta_i$. Therefore $P^*_\omega(\hat{r}_i(\hat{\lambda})) = w_i(P^*_\omega(\hat{\lambda}))$ follows by comparison with (1). Now $\breve{W}$ is the Coxeter group characterized by $(\hat{r}_i \hat{r}_j)^{\breve{m}_{ij}} = 1$. So by induction on the number of generators $\hat{r}_i$ and $\hat{r}_j$ in the expression $(\hat{r}_i \hat{r}_j)^{\breve{m}_{ij}}$, we can deduce from what precedes that

$$P^*_\omega(\hat{\lambda}) = (w_i w_j)^{\breve{m}_{ij}} (P^*_\omega(\hat{\lambda}))\,.$$

Let $w := (w_i w_j)^{\breve{m}_{ij}}$. Since P^*_ω is a bijection between $\hat{H}^*$ and $(H^*)^0$, $w(\lambda) = \lambda$ for all symmetric weights λ in H^*. In particular $w(\rho) = \rho$ as ρ is assumed to be symmetric. The definition of ρ implies that for all $i \in I_{\mathrm{r}}$ (i.e. such that $a_{ii} > 0$), $(\rho, \alpha_i) > 0$. Hence the proof of Proposition 3.12 in [5] tells us that $w = 1$.
We may therefore define a map $\Theta \colon \breve{W} \to \hat{W}$ such that

$$\Theta(\hat{r}_i) = w_i\,,$$

which extends in a natural way to $\breve{W}$. The above reasoning shows that Θ is well defined and a group homomorphism. Proposition 3.3 implies that Θ is surjective.
It only remains to show that Θ is injective. From the preceding calculations we can also deduce that

$$P^*_\omega(\hat{r}(\hat{\lambda})) = \Theta(\hat{r})(P^*_\omega(\hat{\lambda}))\,,$$

for all elements $\hat{r} \in \breve{W}$. So again the bijectivity of P^*_ω implies that if $\Theta(\hat{r}) = 1$, then $\hat{r} = 1$. Thus Θ is a group isomorphism. $\square$

As the next two results show, with respect to the twining character, the subgroup $\hat{W}$ plays the role that the Weyl group plays with respect to the ordinary character:

Proposition 3.5. *If V is an integrable highest weight G-module with highest weight Λ, then $w((\mathrm{ch}\, V)^\omega) = (\mathrm{ch}\, V)^\omega$ for all $w \in \hat{W}$.*

Proof. Let R be the representation: $G \to \mathrm{gl}(V)$, λ a symmetric weight of V, and V_λ the corresponding weight space in V. Then $w_i(\lambda)$ is symmetric, and is a weight of V since Λ is integrable. Set $x^R_l := (\exp f_{\dot{\omega}^l i})(\exp -e_{\dot{\omega}^l i})(\exp f_{\dot{\omega}^l i})$. Define

$$X_i := \begin{cases} x^R_1 x^R_2 \cdots x^R_{N_i - 1}, & \text{if } s_i = 1\,; \\ \prod_{l=1}^{N_i/2} x^R_l x^R_{l+N_i/2} x^R_l\,, & \text{if } s_i = 2\,. \end{cases}$$

Lemma 3.8 in [5] implies that $X_i(V_\lambda) = V_{w_i(\lambda)}$. Since ω extends uniquely to an automorphism of the universal enveloping algebra $U(G)$ of G, the definition of τ_ω implies that

$$\tau_\omega(X_i v) = (\omega^{-1} X_i)\tau_\omega(v) \tag{2}$$

for all $v \in V$. Since w_i commutes with ω^* and the Coxeter relations hold for $\hat{W}$, $\omega^{-1} X_i = X_i$. Hence we can deduce from (2) that the trace of τ_ω on V_λ equals the trace of τ_ω on $V_{w_i(\lambda)}$. Therefore $w_i((\operatorname{ch} V)^\omega) = (\operatorname{ch} V)^\omega$, and the result follows from Proposition 3.3. □

For $w \in \hat{W}$, let $\hat{l}(w)$ be the minimal number of generators w_i needed to write w. Define

$$\hat{\epsilon}(w) := (-1)^{\hat{l}(w)}.$$

Let $\lambda \leq \Lambda$ be symmetric weights in H^*. Consider the Verma module $M(\Lambda)$. Taking a basis of the universal algebra $U(G)$ of G as given by the PBW theorem, we can deduce that the trace of τ_ω on $M(\Lambda)_\lambda$ only depends on the action of ω on $U(G)$. Therefore the expression $e(-\Lambda - \rho)\mathcal{V}_\Lambda^\omega$ is independent of Λ. Set

$$\mathcal{V}^\omega := e(-\Lambda - \rho)\, \mathcal{V}_\Lambda^\omega.$$

Lemma 3.6. *For all $w \in \hat{W}$, $w(\mathcal{V}^\omega) = \hat{\epsilon}(w)\mathcal{V}^\omega$.*

Proof. Consider the Verma module $M(0)$ with highest weight 0. Let $i \in \breve{I}$, and

$$\Delta_i := \{-\beta \in \Delta^+ \mid \exists\, j \neq \dot{\omega}^l i \ \ \forall l \in \mathbb{Z},\ \text{such that } \alpha_j \leq \beta\}\,.$$

The ordinary character of $M(0)$ is $\mathcal{V}_0 = \prod_{\alpha \in \Delta^+}(1 - e(-\alpha))^{-\operatorname{mult}\alpha}$. Therefore all weights $\mu \in H^*$ such that $\mu \leq 0$ are weights of $M(0)$. Let $\lambda \leq 0$ be a symmetric weight. We can write $\lambda = \sum_{\beta \in \Delta_i} \beta - n \sum_{l=0}^{N_i - 1} \alpha_{\dot{\omega}^l i}$ for some non-negative integer n.

When $s_i = 1$, then for all integers l, l', $\alpha_{\dot{\omega}^l i} + \alpha_{\dot{\omega}^{l'} i}$ is not a root. So we can order the positive roots of G in the following way: $\alpha_i, \alpha_{\dot{\omega} i}, \ldots, \alpha_{\dot{\omega}^{N_i - 1} i}, \gamma_1, \gamma_2, \ldots$, with $\gamma_p \in \Delta_i$.

When $s_i = 2$, then for $l, l' \in \mathbb{Z}$, $\alpha_{\dot{\omega}^l i} + \alpha_{\dot{\omega}^{l'} i}$ is a root if and only if $l' \equiv l + \frac{N_i}{2} \pmod{N_i}$; and for all $l, l', l'' \in \mathbb{Z}$, $\alpha_{\dot{\omega}^l i} + \alpha_{\dot{\omega}^{l'} i} + \alpha_{\dot{\omega}^{l''} i}$ is not a root. In this case, we can order the positive roots of G in the following way: $\alpha_i, \alpha_{\dot{\omega} i}, \ldots, \alpha_{\dot{\omega}^{N_i - 1} i}, \alpha_i + \alpha_{\dot{\omega}^{N_i/2} i}, \ldots, \alpha_{\dot{\omega}^{N_i/2 - 1} i} + \alpha_{\dot{\omega}^{N_i - 1} i}, \gamma_1, \gamma_2, \ldots$, with $\gamma_p \in \Delta_i$.

By definition, $\tau_\omega(xv) = \omega^{-1}(x)\tau_\omega(v)$ for all $x \in U(G)$ and all $v \in M(0)$. So choosing a basis of $M(0)_\lambda$ given by the PBW theorem, which respects the above ordering of roots, we can deduce that the only basis vectors of $M(0)_\lambda$ contributing to the trace of τ_ω are as follows:

For the case $s_i = 1$, these vectors are

$$f_i^m f_{\dot{\omega} i}^m \cdots f_{\dot{\omega}^{N_i - 1} i}^m v_q^{(m)}\,,$$

where $0 \leq m \leq n$ and the vectors $v_q^{(m)}$ form a basis of the weight space $M(0)_{\lambda + m\beta_i}$;

and for the case $s_i = 2$, if $\tilde{f}_i := [f_i, f_{\dot\omega^{N_i/2} i}]$, these vectors are

$$f_i^{m_0} f_{\dot\omega i}^{m_1} \cdots f_{\dot\omega^{N_i-1} i}^{m_{N_i-1}} \tilde{f}_i^{n_0} \tilde{f}_{\dot\omega i}^{n_1} \cdots \tilde{f}_{\dot\omega^{N_i/2-1} i}^{n_{N_i/2-1}} v_q^{(m_k,n_j)},$$

where $m_k = m_{k+N_i/2}$, $m_k + n_k = m_j + n_j \le n$ for all $0 \le j,k \le N_i/2-1$, and the vectors $v_q^{(m_k,n_j)}$ form a basis of the weight space $M(0)_{\lambda+(m_0+n_0)\beta_i}$.

Commutator terms that arise when reshuffling the products of the generators of the corresponding root spaces to the form given by the chosen basis can never give rise to a non-zero contribution to the trace of τ_ω in $M(0)_\lambda$. Therefore if $m_\lambda^\omega := \mathrm{tr}(\tau_\omega)|_{M(0)_\lambda}$, then summing over all the symmetric weight spaces of $M(0)$, we can deduce that

$$\mathcal{V}_0^\omega = \Big(\sum_\lambda m_\lambda^\omega e(\lambda)\Big)\,(1 + m_{-\beta_i}^\omega e(-\beta_i) + m_{-2\beta_i}^\omega e(-2\beta_i) + \dots)\,, \tag{3}$$

where the first sum is taken over all sums of roots in Δ_i.

From the above we can also deduce that for $s_i = 1$,

$$\mathrm{tr}(\tau_\omega)_{M(0)_{-n\beta_i}} = 1$$

since all the simple roots $\alpha_{\dot\omega^l i}$ are pairwise orthogonal; and for $s_i = 2$,

$$\mathrm{tr}(\tau_\omega)_{M(0)_{-n\beta_i}} = \sum_{\substack{0\le n_j\le n\\ 0\le j\le N_i/2-1}} (-1)^{\sum_{j=0}^{N_i/2-1} n_j} = \Big(\sum_{k=0}^{n}(-1)^k\Big)^{N_i/2-1},$$

so that

$$\mathrm{tr}(\tau_\omega)_{M(0)_{-n\beta_i}} = \begin{cases} 1\,, & \text{if } n \equiv 0 \pmod 2\,,\\ 0\,, & \text{otherwise}\,. \end{cases}$$

Substituting the value of $\mathrm{tr}(\tau_\omega)_{M(0)_{-n\beta_i}}$ in (3), we get

$$\mathcal{V}_0^\omega = \Big(\sum_{\lambda\in\Delta_i} m_\lambda^\omega e(\lambda)\Big)\,(1 - e(-s_i\beta_i))^{-1}\,.$$

Since ρ is symmetric by assumption, and $(\rho,\alpha_i) = \frac{1}{2}(\alpha_i,\alpha_i)$ for all $i\in I$, (1) gives: $w_i(\rho) = \rho - s_i\beta_i$. Now r_i permutes the set of all negative roots distinct from $-\alpha_i$. Therefore w_i permutes the elements of Δ_i and multiplies $e(-\rho)\,(1 - e(-s_i\beta_i))^{-1}$ by -1. Hence the assertion of the Lemma follows for the generator w_i. This argument works for each $i\in\breve{I}_{\mathrm{r}}$, and so Proposition 3.3 implies the Lemma for all $w\in\hat{W}$. $\square$

Set

$$B_\Lambda^\omega := \{\lambda\in H^* \mid \lambda \le \Lambda,\ \big|\lambda+\rho\big| = \big|\Lambda+\rho\big|,\ \omega^*(\lambda) = \lambda\}.$$

Arguments similar to those used in [2] or in §9 of [5] imply that for any symmetric weight Λ in H^*,

$$\mathcal{V}_\Lambda^\omega = \sum_{\lambda\in B_\Lambda^\omega} c_{\Lambda\lambda}\,\Psi_\lambda^\omega,$$

where $c_{\Lambda\lambda} \in \mathbb{C}$, and $c_{\Lambda\Lambda} = 1$. Since B_Λ^ω is a discrete set, by inverting the upper triangular matrix $(c_{\Lambda\lambda})_{\lambda \in B_\Lambda^\omega}$, we get

$$\Psi_\Lambda^\omega = \sum_{\lambda \in B_\Lambda^\omega} c_\lambda \, \mathcal{V}_\lambda^\omega, \tag{4}$$

where $c_\Lambda = 1$.

Lemma 3.7. *Suppose that $\Lambda \in H^*$ is a symmetric integrable weight. Then for each scalar c_λ in equation (4), there is some $w \in \hat{W}$ such that $c_\lambda = \hat{\epsilon}(w) c_{w(\Lambda-\alpha+\rho)-\rho}$, where α is a symmetric sum of distinct pairwise orthogonal imaginary simple roots, all orthogonal to Λ.*

Proof. From (4) we get

$$\Psi_\Lambda^\omega = \mathcal{V}^\omega \sum_{\lambda \in B_\Lambda^\omega} c_\lambda \, e(\lambda + \rho)\,, \tag{5}$$

where $\mathcal{V}^\omega$ is independent of λ. Given $\lambda \in B_\Lambda^\omega$, let $w \in \hat{W}$ be such that the height of $\Lambda + \rho - w(\lambda + \rho)$ is minimal. Let $\mu := w(\lambda + \rho) - \rho$. The proof of Proposition 3.3 shows that $(\lambda + \rho, \alpha_i) \geq 0$ for all real simple roots α_i. Then $\mu = \Lambda - \sum_{i \in I} k_i \alpha_i$, where the k_i are non-negative integers. Furthermore $|\mu + \rho|^2 = |\Lambda + \rho|^2$ implies that

$$\sum_{i \in I} k_i(\Lambda, \alpha_i) + \sum_{i \in I} k_i(\mu + 2\rho, \alpha_i) = 0\,.$$

So as in the proof of Theorem 11.13.3 in [5] it follows that if $k_i \neq 0$ then α_i is imaginary and $(\alpha_i, \Lambda) = 0$; that $(\alpha_i, \alpha_j) = 0$ if k_j is also non-zero for $j \neq i$; and that $(\alpha_i, \alpha_i) = 0$ if $k_i \geq 2$. Since Ψ_Λ^ω is bounded by the ordinary character Ψ_Λ, terms such as $e(\Lambda - \sum_{i \in I} k_i \alpha_i)$ do not occur in Ψ_Λ^ω as all the roots α_i are orthogonal to Λ (see §11 of [5]). The ordinary character of the Verma module $M(\Lambda)$ equals $e(\Lambda) \prod_{\alpha \in \Delta^+} (1 - e(-\alpha))^{-\mathrm{mult}\,\alpha}$, so that $\mathcal{V}^\omega$ is bounded by $e(-\rho) \prod_{\alpha \in \Delta^+} (1 - e(-\alpha))^{-\mathrm{mult}\,\alpha}$. Hence we can deduce that if $k_i \neq 0$, then $k_i = 1$. The Lemma now follows from Proposition 3.5 and Lemma 3.6. $\square$

We next determine the values of the scalars c_λ in (5) when $\Lambda - \lambda$ is a sum of imaginary simple roots, pairwise orthogonal, and all orthogonal to Λ. Note that since Λ and λ are both symmetric, so is $\Lambda - \lambda$. Hence $\lambda = \Lambda - \sum_{i \in \breve{I}} \beta_i$, where the sum can be taken to be over $\breve{I}$ rather than the larger $\hat{I}$, as $(\alpha_i, \alpha_{\dot{\omega}^l i}) = 0$ for all integers $1 \leq l \leq N_i - 1$ implies that i is $\dot{\omega}$-conjugate to an integer in $\breve{I}$.

We first need another definition. If $\gamma = \sum_{i \in \breve{I}} k_i \beta_i$, define

$$\breve{\mathrm{ht}}(\gamma) := \sum_{i \in \breve{I}} k_i\,.$$

Lemma 3.8. *Let Λ be a symmetric integrable weight in H^*, and λ be a symmetric element in B_Λ^ω. If $\Lambda - \lambda$ is a sum of distinct, pairwise orthogonal, imaginary simple roots, all orthogonal to Λ, then the coefficient c_λ in (4) is*

$$c_\lambda = (-1)^{\breve{\mathrm{ht}}(\Lambda - \lambda)}.$$

Proof. We prove this Lemma by induction on $\breve{\mathrm{ht}}(\Lambda-\lambda)$. We know from (4) that $c_\Lambda = 1$. Set C_Λ to be the set of all symmetric weights μ in B^ω_Λ such that $\Lambda-\mu$ is the sum of distinct, pairwise orthogonal, imaginary simple roots, and let

$$\mathcal{V}^\omega_0 = \sum m^\omega_\mu \, e(\mu) \, .$$

Since the ordinary character of the Verma module $M(0)$ of G is

$$\mathcal{V}_0 = \prod_{\alpha\in\Delta^+} (1-e(-\alpha))^{-\mathrm{mult}\alpha} \, ,$$

for all μ in C_0 the dimension of the weight space $M(0)_\mu$ is 1, as $-\mu$ is a sum of orthogonal simple roots. So the definition of τ_ω gives $m^\omega_\mu = 1$ for all $\mu \in C_0$. Let $\lambda \in C_\Lambda$, and so $\Lambda - \lambda = \sum_{s=1}^r \beta_{i_s}$, where the β_{i_s} are all distinct. The coefficient of $e(\lambda)$ on the right hand side of (4) is

$$\sum_{s=0}^r \sum_{\{j_1,\ldots,j_s\}\in T_s} c_{\Lambda-\beta_{i_{j_1}}-\ldots-\beta_{i_{j_s}}} \, ,$$

where T_s is the set of all subsets of $\{1,2,\ldots,r\}$ of order s. Since $e(\lambda)$ does not appear on the left hand side of (4), this coefficient equals 0. Assume now that the result holds for all weights μ in C_Λ such that $\breve{\mathrm{ht}}(\Lambda-\mu) < \breve{\mathrm{ht}}(\Lambda-\lambda)$. It follows by induction that

$$c_\lambda + (-1)^{\breve{\mathrm{ht}}(\Lambda-\lambda)} \sum_{s=1}^r \binom{r}{s} (-1)^s = 0 \, ,$$

which gives the desired answer for c_λ. $\square$

Combining the results of Lemma 3.7 and Lemma 3.8 we have therefore proved that when Λ is a symmetric integrable weight, then

$$\Psi^\omega_\Lambda = \mathcal{V}^\omega \sum_{w\in\hat{W}} \hat{\epsilon}(w) \, w(S^\omega_\Lambda) \, ,$$

where

$$S^\omega_\Lambda = e(\Lambda+\rho) \sum \hat{\epsilon}(\beta) \, e(-\beta) \, ,$$

and $\hat{\epsilon}(\beta) = (-1)^{\breve{\mathrm{ht}}(\beta)}$ if β is the symmetric sum of pairwise orthogonal imaginary simple roots, all orthogonal to Λ, and $\hat{\epsilon}(\beta) = 0$ otherwise. Also of course, for the trivial module $\Psi^\omega_0 = 1$; we can therefore deduce that

$$\mathcal{V}^\omega = \Big(\sum_{w\in\hat{W}} \hat{\epsilon}(w) w(S^\omega_0) \Big)^{-1} \, .$$

Substituting this result back into the formula for Ψ^ω_Λ, we obtain for any integrable symmetric weight Λ in H^*,

$$\Psi^\omega_\Lambda = \frac{\sum_{w\in\hat{W}} \hat{\epsilon}(w)\, w(S^\omega_\Lambda)}{\sum_{w\in\hat{W}} \hat{\epsilon}(w)\, w(S^\omega_0)},$$

and for any symmetric weight Λ in H^*

$$\mathcal{V}^\omega_\Lambda = e(\Lambda)\Big(\sum_{w\in\hat{W}} \hat{\epsilon}(w) w(S^\omega_0)\Big)^{-1}.$$

We can now complete the Proof of Theorem 3.1.

Proof of Theorem 3.1. Let $i \in \hat{I}$ and α_i be imaginary, then $(\alpha_i, \omega^{*l}(\alpha_i)) = 0$ for all integers $1 \le l \le N_i - 1$ if and only if $i \in \breve{I}$ and $\hat{\alpha}_i$ is an imaginary root of $\breve{G}$.
When Λ is an integrable symmetric weight in H^*, it follows from Lemma 2.3 that $P^{*-1}_\omega(\Lambda)$ is an integrable weight in $\hat{H}^*$ for the GKM algebra $\breve{G}$ (note that it is not integrable for the bigger algebra $\hat{G}$). Furthermore Corollary 3.4 implies that the minimal number of generators w_i of $w \in \hat{W}$ equals the number of generators $\hat{r}_i$ of $\Theta^{-1}(w)$ in $\breve{W}$. Therefore the ordinary character of the irreducible $\breve{G}$-module of highest weight $P^{*-1}_\omega(\Lambda)$ equals $P^{*-1}_\omega \Psi^\omega_\Lambda$, when Λ is integrable; and for any symmetric weight Λ, the character of the Verma $\breve{G}$-module of highest weight $P^{*-1}_\omega(\Lambda)$ equals $P^{*-1}_\omega \mathcal{V}^\omega_\Lambda$. This completes the proof of Theorem 3.1. $\square$

It follows that if V is a highest weight G-module with symmetric highest weight Λ, and

$$(\mathrm{ch}\, V)^\omega = \sum_{\lambda\le\Lambda} m^\omega_\lambda\, e(\lambda),$$

then $m^\omega_\lambda \neq 0$ implies that $\Lambda - \lambda = \sum_{i\in\breve{I}} k_i\beta_i$, where for all $i \in \breve{I}$, k_i is a non-negative integer. Note that the sum may be taken to be over $\breve{I}$, and not the larger $\hat{I}$. The denominator formula for $\breve{G}$ immediately gives the following Corollary.

Corollary 3.9. *Let $\breve{\Delta}^+$ denote the set of positive roots of $\breve{G}$. If Λ is a symmetric weight in H^*, then $\mathcal{V}^\omega_\Lambda = e(\Lambda)\prod_{\hat{\alpha}\in\breve{\Delta}^+}(1 - e(P^*_\omega(-\hat{\alpha})))^{-\mathrm{mult}\hat{\alpha}}$, where $\mathrm{mult}\,\hat{\alpha} = \dim \breve{G}_\alpha$.*

Remarks. 1. When $\hat{G} \neq \breve{G}$, the twining characters for highest weight G-modules coincide with ordinary characters of $\breve{G}$ and not of $\hat{G}$. This is due to the fact that when $\hat{I} \neq \breve{I}$, $P^{*-1}_\omega(\rho)$ is not always a Weyl vector for $\hat{G}$.
2. If $\breve{I} = \emptyset$, the above results implies that for all symmetric weights Λ in H^*, $\mathcal{V}^\omega_\Lambda = e(\Lambda)$, and $\Psi^\omega_\Lambda = e(\Lambda)$ if Λ is also integrable. In this case $\breve{G} = 0$, and $P^{*-1}_\omega(\Lambda) = 0$, so that $P^{*-1}_\omega(\mathcal{V}^\omega_\Lambda) = e(0)$ and $P^{*-1}_\omega \Psi^\omega_\Lambda = e(0)$. This is Theorem 2 in [2].
3. Since for fixed Cartan matrix, the ordinary character and the twining character do not depend on the size of the Cartan subalgebra, Theorem 3.1 holds for any GKM algebras $G = G(A)$ and $\breve{G} = G(\breve{A})$ as long as the duals of their Cartan subalgebras are large enough

for all roots to be either positive or negative, for the multiplicities of the simple roots to be finite, and for the existence of Weyl vectors. In particular the bilinear forms induced by A and $\breve{A}$ need not be non-degenerate, and the multiplicities of the simple roots may be greater than 1.

4. A larger class of outer automorphisms.

As before the Cartan matrix A is symmetric, $G = G(A)$ denotes a GKM algebra such that the bilinear form on G induced by A is non-degenerate; and the bijection $\dot{\omega}$ of the set I preserves the Cartan matrix A. Also, ω denotes the outer automorphism of G defined in section 2. To the bijection $\dot{\omega}$ we can in fact not only associate the automorphism ω, but a whole family of outer automorphisms of G. More precisely, there exist automorphisms $\tilde{\omega}$ of G such that

$$\tilde{\omega} e_i = \xi_i e_{\dot{\omega} i} \,, \quad \tilde{\omega} f_i = \xi_i' f_{\dot{\omega} i} \,,$$

where ξ_i, and ξ_i' are in $\mathbb{C}^*$. (The proof of the existence of ω in [2] applies to $\tilde{\omega}$ as well). It follows immediately that

$$\tilde{\omega} = \phi \omega \,,$$

where ϕ is an inner automorphism of G such that

$$\phi e_i = \xi_i e_i \,, \quad \phi f_i = \xi_i' f_i \,.$$

We will refer to the automorphisms ω and $\tilde{\omega}$ as diagram automorphisms and generalized diagram automorphisms, respectively.

Lemma 4.1. *With the above notation, $\xi_i' = \xi_i^{-1}$ for all $i \in I$ for which there exists $j \in I$ such that $a_{ij} \neq 0$.*

Proof. Since ϕ is a Lie algebra homomorphism, $\phi(h_i) = \xi_i \xi_i' h_i$. Now on the one hand, for all $j \in I$, $\phi([h_i, e_j]) = \phi(a_{ij} e_j) = a_{ij} \phi(e_j)$. On the other hand, $[\phi(h_i), \phi(e_j)] = \xi_i \xi_i' a_{ij} \phi(e_j)$. Hence $a_{ij} \phi(e_j) = \xi_i \xi_i' a_{ij} \phi(e_j)$, and the result follows. $\square$

We now assume that for all $i \in I$, $\xi_i' = \xi_i^{-1}$. Therefore ω and $\tilde{\omega}$ are equal on the Cartan subalgebra H, so that $\omega^* = \tilde{\omega}^*$ (i.e. the dual of $\tilde{\omega}|_H$ on H^*). As in the previous section, $\tilde{\omega}$ induces a bijection $\tau_{\tilde{\omega}}$ of G-modules: $\tau_{\tilde{\omega}} : V_\Lambda \to V_{\omega^*(\Lambda)}$, which satisfies the *$\tilde{\omega}$-twining property*, i.e.

$$\tau_{\tilde{\omega}}(R_\Lambda(x) v) = R_{\tilde{\omega}^*(\Lambda)}(\tilde{\omega}^{-1} x) \tau_{\tilde{\omega}}(v)$$

for all $x \in G$ and all $v \in V_\Lambda$. When $\omega^*(\Lambda) = \Lambda$, we define the twining character of a G-module V with respect to $\tilde{\omega}$ to be

$$(\mathrm{ch}\, V)^{\tilde{\omega}}(h) = \mathrm{tr}_V \tau_{\tilde{\omega}} \mathrm{e}^{R_\Lambda(h)}.$$

We next show that $(\mathrm{ch}\, V)^{\tilde{\omega}}$ can be easily expressed in terms of $(\mathrm{ch}\, V)^{\omega}$.

Note that unlike in section 3, we now require the bilinear form on H to be non-degenerate.

Theorem 4.2. *Let $\Lambda \in H^*$ be such that $\omega^*(\Lambda) = \Lambda$. There exists an element $h_{\tilde\omega} \in H^0$ such that the twining character of the Verma module of highest weight Λ with respect to $\tilde\omega$ is*

$$\mathcal{V}_\Lambda^{\tilde\omega}(h) = e(\Lambda(h_{\tilde\omega}))\, \mathcal{V}_\Lambda^{\omega}(h - h_{\tilde\omega})\,, \quad h \in H\,.$$

If moreover, Λ is also integrable, then the twining character of the irreducible module of highest weight Λ with respect to $\tilde\omega$ is

$$\Psi_\Lambda^{\tilde\omega}(h) = e(\Lambda(h_{\tilde\omega}))\, \Psi_\Lambda^{\omega}(h - h_{\tilde\omega})\,, \quad h \in H\,.$$

Proof. We write the irreducible twining characters for the diagram automorphism ω as

$$\Psi_\Lambda^{\omega} = \sum_{\lambda \le \Lambda} m_\lambda^{\omega}\, e(\lambda) \tag{6}$$

and for the generalized diagram automorphism as

$$\Psi_\Lambda^{\tilde\omega} = \sum_{\lambda \le \Lambda} m_\lambda^{\tilde\omega}\, e(\lambda)\,. \tag{7}$$

Let λ be an element in H^* such that $m_\lambda^\omega \ne 0$ (or equivalently, $m_\lambda^{\tilde\omega} \ne 0$). Then $\lambda = \Lambda - \sum_{i\in I} k_i\alpha_i$, where for each $i \in I$, k_i is a non-negative integer, $k_{\dot\omega i} = k_i$, and $k_i = 0$ unless i is $\dot\omega$-conjugate to an element in $\check I$. We find that

$$m_\lambda^{\tilde\omega} = m_\lambda^{\omega} \prod_{i\in I} (\xi_i)^{k_i} = m_\lambda^{\omega} \prod_{i\in \check I} \Big(\prod_{l=0}^{N_i-1} \xi_{\dot\omega^l i} \Big)^{k_i}\,.$$

For each $i \in I$, we define $\sigma_i \in \mathbb{C}$ to be as follows:

$$\prod_{l=0}^{N_i-1} \xi_{\dot\omega^l i} = \mathrm{e}^{\sigma_i/N_i}\,.$$

The imaginary part of σ_i is of course only fixed modulo $2\pi N_i$, and we may put $\sigma_{\dot\omega i} = \sigma_i$ for all $i \in I$. This allows us to express $m_\lambda^{\tilde\omega}$ as

$$m_\lambda^{\tilde\omega} = m_\lambda^{\omega} \prod_{i\in \check I} \mathrm{e}^{\sigma_i k_i/N_i} = m_\lambda^{\omega} \prod_{i\in I} \mathrm{e}^{\sigma_i k_i}\,.$$

For $i \in I$, let Λ_i denote the fundamental weights, satisfying $(\Lambda_i, \alpha_j) = \delta_{ij}$ for all $i, j \in I$. We can write k_i as $k_i = (\Lambda - \lambda, \Lambda_i)$. Define the element

$$\sigma := \sum_i \sigma_i \Lambda_i\,.$$

Then σ is in $(H^*)^0$. We obtain

$$m_\lambda^{\tilde\omega} = m_\lambda^{\omega}\, \mathrm{e}^{(\Lambda-\lambda,\sigma)} \,.$$

Let $\varphi:\ H^* \to H$ be the bijection induced by the bilinear form $(.\,,.)$ on H, and let $h_{\tilde\omega}$ denote $\varphi(\sigma)$. Substituting the above expression in (7) and using (6) we find that $\Psi_\Lambda^{\tilde\omega}(h) = e(\Lambda(h_{\tilde\omega}))\Psi_\Lambda^{\omega}(h-h_{\tilde\omega})$, proving the Theorem. $\square$

Hence the effect of a generalized diagram automorphism, as compared to the corresponding ordinary diagram automorphism, consists in a shift in the argument and a multiplication by an overall factor. In case $\tilde\omega$ has finite order, this factor is of course a phase. Note that the imaginary part of σ is defined only up to 2π times an element of the weight lattice of G; the real part of σ is unique, however; it is zero if $\tilde\omega$ has finite order.

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